EXERCISE – IV

HINTS & SOLUTIONS

Sol.1 **(i)** $|\alpha + \beta|^2 + |\alpha + \beta|^2$

$= (\alpha + \beta)(\bar{\alpha} + \bar{\beta}) + (\alpha - \beta)(\bar{\alpha} - \bar{\beta})$

$= \alpha\bar{\alpha} + \beta\bar{\beta}$

$= 2(|\alpha|^2 + |\beta|^2)$

(ii) Let $z_1 = \alpha + \sqrt{\alpha^2 - \beta^2}$: $z_2 = \alpha - \sqrt{\alpha^2 - \beta^2}$

$z_1 + z_2 = 2\alpha$

$z_1 - z_2 = 2\sqrt{\alpha^2 - \beta^2}$

$z_1 z_2 = \beta^2$

$|z_1| + |z_2| = |\alpha + \beta| + |\alpha - \beta|$

Squaring on both the side

$|z_1|^2 + |z_2|^2 + 2|z_1 z_2| = [|\alpha + \beta| + |\alpha - \beta|]^2$

as we know that $2|z_1|^2 + 2|z_2|^2 = |z_1 + z_2|^2 + |z_1 - z_2|^2$

LHS $= \frac{1}{2}[|z_1 + z_2|^2 + |z_1 - z_2|^2 + 2|z_1 z_2|]$

$= \frac{1}{2}[|2\alpha|^2 + 4|\alpha^2 - \beta^2| + 2|\beta^2|]$

$= [2|\alpha|^2 + 2|\beta|^2 + 2|\alpha + \beta||\alpha - \beta|]$

$= |\alpha + \beta|^2 + |\alpha - \beta|^2 + 2|\alpha + \beta||\alpha - \beta|$

$= [|\alpha + \beta| + |\alpha - \beta|]^2$ = RHS

Sol.2 **(a)** $(1 + w)^7 = A + B\omega$

w is cube root of unety

so $1 + w + w^2 = 0$

$\mu 1 + \omega = -\omega^2$

$(-\omega^2)^7 = A + B\omega$

$-\omega^{14} = A + B\omega$

$-\omega^2 = A + B\omega$

$(1 + \omega) = A + B\omega$

urdeced pair

(A, B) = (1, 1)

(b) $S = 1^3 - 1 + 2^3 - 1 + 3^3 - 1 \ldots\ldots + n^3 - 1$

$S = \left[\frac{n(n+1)}{2}\right]^2 - n$

Sol.3 **(a)** put z = x

$x^2 - (3 + i)x + m + 2i = 0$

$(x^2 - 3x + m) + i(2 - x) = 0$

$x^2 - 3x + m = 0$ & $2 - x = 0 \Rightarrow x = 2$

put x = 2

m = 2

(b) $P(z) = 2z^4 + cz^3 + bz^2 + cy + 3$

a,b,c one real

two roots one 2 and i

so theirel root will be – i Let 4th root is = α

ploduct of root $= 2 \times i \times -i \times a = -\frac{3}{2}$

$\alpha = 3/4$

sum of root

$2 + i - i + 3/4 = -1/2$

$11/4 = -a/2$

$a = -1/2$

Sol.4 **(i)** $z = 1 + \cos\frac{10\pi}{9} + i\sin\left(\frac{10\pi}{9}\right)$

$z = 2\cos^2\frac{5\pi}{9} + i\,2\sin\frac{5\pi}{9}.\cos\frac{5\pi}{9}$

$z = 2\cos\frac{5\pi}{9}\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right)$

modulus $= 2\cos\frac{5\pi}{9}$

principal arg (z) $= \frac{5\pi}{9} = -\frac{4\pi}{9}$

$\arg(z) = 2k\pi - \frac{4\pi}{9}$; $k \in I$

(ii) $(\tan 1 - i)^2$

$\tan^2 1 - 1 - 2i\tan 1$

$= \frac{1}{\cos^2 1}(-\cos 2 - i\sin 2)$

Modulus = $\sec^2 1$

arg $= 2n\pi + (2 - \pi)$

Principal arg $= (2 - \pi)$

(iii) $z = \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-2i}}$

$\sqrt{5+12i} = x + iy$

$5 + 12i = x^2 - y^2 + 2ixy$

$x^2 - y^2 = 5$

$4x^2y^2 = 12$

$(x^2 + y^2)^2 = 25 + 144$

$x^2 + y^2 = 13$

$x^2 = 9$ $x = \pm 3$

$y = \pm 2$

$\sqrt{5+12i} = 3 + 2i$ or $-3 - 2i$

$\sqrt{5+12i} = 3 - 2i$

So $z = \frac{6}{4i} = \frac{3}{2i} = -\frac{3}{2}i$

$\arg z = \frac{-\pi}{2}$ Principal value of arg $= \frac{\pi}{2}$

$|z| = 3/2$

(iv) $z = \dfrac{i1}{i(1-\cos\pi)+\sin\pi}$

$z = \dfrac{i}{2i} = \dfrac{1}{2}$

modulus = $\dfrac{1}{2}$

Principal Arg (z) = 0

Sol.5 $\displaystyle\sum_{k=1}^{2n}\left(\sin\frac{2\pi k}{2n+1} - i\cos\frac{2\pi k}{2n+1}\right)$

$\displaystyle\sum_{k=1}^{2n} -i\left(\cos\frac{2\pi k}{2n+1} + i\sin\frac{2\pi k}{2n+1}\right)$

$z^{2n+1} = 1$

$z^{2n+1} - 1 = 0$

$z = \left(\cos\dfrac{2\pi k}{2n+1} + i\sin\dfrac{2\pi k}{2n+1}\right)$

put k = 0, 1, 2, ...m

$z_0 = 1$

$z_0 + z_1 + ...z_{2n} = 0$

$z_0 + \displaystyle\sum_{k=1}^{2n}\cos\frac{2\pi k}{2n+1} + i\ \sin\frac{2\pi k}{2n+1} = 0$

$\displaystyle\sum_{k=1}^{2n}\cos\frac{2\pi k}{2n+1} + i\ \sin\frac{2\pi k}{2n+1} = -1$

$-i\displaystyle\sum_{k=1}^{2n}\cos\frac{2\pi k}{2n+1} + i\ \sin\frac{2\pi k}{2n+1} = i$

Sol.6 Assume $\dfrac{1+i}{2} = z$; multiply numerator and denominator by (1 – z) which simplifies to

$= \dfrac{1-\left(z^2\right)^{2^n}}{1-z}$; Now $\dfrac{1}{1-z} = \dfrac{2}{1-i} = (1+i)$

$\left(z^{2^n}\right)^2 = \left(z^2\right)^{2^n} = \left[\left(\dfrac{1+i}{2}\right)^2\right]^{2^n} = \left(\dfrac{i}{2}\right)^{2^n}$

for $n \ge 2$ $(i)^{2^n} = 1 \Rightarrow \left(z^{2^n}\right)^2 = \dfrac{1}{2^{2^n}}$

$\Rightarrow$ Given expression = $\left(1 - \dfrac{1}{2^{2^n}}\right)(1+i)$

Sol.7 **(a)** $\text{Re}\left(\dfrac{z+2i}{iz+2}\right) \le 4$

$z = x + iy$

$\text{Re}\left(\dfrac{x+(y+2)i}{i(x+iy)+2}\right)$

$\text{Re}\left(\dfrac{x+(y+2)i}{(2-y)+ix} \times \dfrac{(2-y)-ik}{(2-y)-ik}\right)$

$= \text{Re}\left(\dfrac{x+(y+2)+x(2-y)+i(y+2)(2-y)-ix(y+2)i}{(y-2)^2+k^2}\right)$

$= \dfrac{x(y+2)+x(2-y)}{(y-2)^2+x^2} \le 4$

$\dfrac{4x}{(y-2)^2+k^2} \le 4$

$x < (y-2)^2 + k^2$

$x^2 - x + (y-2)^2 \ge 0$

Region outside or on the circle with centre 1/2 + 2i radius 1/4

(b) arg (z + i) – arg (z – i) = H/2

$\arg\left(\dfrac{z+i}{z-i}\right) = H/2$

$x^2 + y^2 = 1$

Sol.8 $z_1^2 + z_2^2 + z_1z_2 = 0$

$\Rightarrow z_2 = z_1\, e^{i\left(\frac{2\pi}{3}\right)} \Rightarrow |z_1| = |z_2|$

angle between z_1 and z_2 is $\dfrac{2\pi}{3}$

Sol.9 $z = (aw+b)(w-c)^{-1}$

$z = \dfrac{aw+b}{w-c}$

$w = \dfrac{b+zc}{z-a}$

$|w| = \left|\dfrac{b+zc}{z-a}\right| = 1$

$\Rightarrow (1-1)^2|z|^2 - 2(a+b+c)\,\text{Re}(z) + a^2 - b^2 = 0$

Sol.10 **(a)** use $C_1 \to C_2 + C_3 - C_1$

(b) $D = e^{-i(A+B+C)} \begin{vmatrix} e^{-iA} & e^{i(A+C)} & e^{i(B+A)} \\ e^{i(B+C)} & e^{-iB} & e^{i(A+B)} \\ e^{i(B+C)} & e^{i(A+C)} & e^{-iC} \end{vmatrix}$

$$= -\begin{vmatrix} e^{-iA} & -e^{iB} & -e^{-iC} \\ -e^{iA} & e^{-iB} & -e^{-iC} \\ -e^{-iA} & -e^{-iB} & e^{-iC} \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

Sol.11 **(a)** $(1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8)$ ..... to 2n factors

$1 + w^2 = -w$

$(-2w)(-2w^2)(-2w)(-2w^2)$.....

$= 4.\ 4.\ 4.\ .....4 = 2^{2n}$

(b) $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$..... upon n factors

$= (-w^2)(-w)(-w^2)(-w)$ ....

$$= \begin{Bmatrix} 1 & \text{if } n \text{ is even} \\ -w^2 & \text{if } n \text{ is odd} \end{Bmatrix}$$

Sol.12 $\text{LHS} = \left[\dfrac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)}\right]^n$

$= \left[\dfrac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\right]^n$ where $\alpha = \dfrac{\pi}{2} - \theta$

$= \left[\dfrac{2\cos^2\alpha/2+2i\sin\alpha/2\,\cos\alpha/2}{2\cos^2\alpha/2-2i\sin\alpha/2\,\cos\alpha/2}\right]^n$

$= \left[\dfrac{e^{i\alpha/2}}{e^{-i\alpha/2}}\right]^n = e^{in\alpha}$

$= \cos n\alpha + i \sin n\alpha$

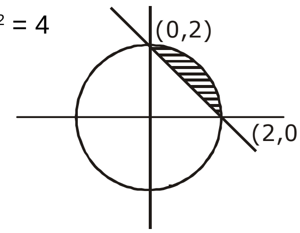
$= \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i \sin\left(\dfrac{n\pi}{2} - n\theta\right)$

Now $\left(\dfrac{1+\sin\pi/5+i\cos\pi/5}{1+\sin\pi/5-i\cos\pi/5}\right)^5 + i = 0$

LHS $\cos\left(\dfrac{5\pi}{2} - 5\times\dfrac{\pi}{5}\right) + i\sin\left(\dfrac{5\pi}{2} - 5\times\dfrac{\pi}{5}\right) + i$

$= 0 - i + i = 0$ = RHS

Sol.13 **(a)**



$A = |z| \le 2 \Rightarrow x^2 + y^2 = 4$

$B = x + y - 2 \ge 0$

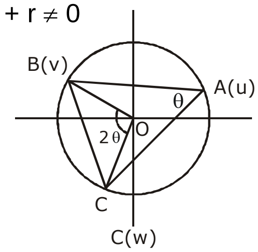
Area $= \dfrac{1}{4}\pi(4) - 2$

$= \pi - 2$

Sol.14 $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$\Rightarrow \dfrac{1}{2}(p + q + r)[(p - q)^2 + (q - r)^2 + (r - p)^2] = 0$

$\Rightarrow p = q = r$ as $p + q + r \ne 0$



$|u| = p$

$|v| = q$

$|w| = r$

$\arg\left(\dfrac{w}{v}\right) = \angle BOC$

$= 2\arg\left(\dfrac{w-u}{v-u}\right) = \arg\left(\dfrac{w-u}{v-u}\right)^2$

Sol.15 $x^3 = 9 + 46i$

$(9 + 46i)^{1/3} = a + ib$

$9 + 46i = (a + ib)^3$ .....(1)

$9 + 46i = (a^3 + 3a^2b) - i(3ab^2 + b^3)$

$\Rightarrow a^3 + 3a^2 b = 9$

$\Rightarrow b^3 + 3ab^2 = -46$

$(a + b)^3 = -37$ .....(2)

& take modulus on equation (1) the side

$|9 + 46i| = |a + ib|^3$

$2197 = (a^2 + b^2)^3$ .....(3)

Solve (2) & (3) and get the value.

Sol.17 $|z - 1| = 1$

$z - 1 = \cos\theta + i\sin\theta$

$z - 2 = \cos\theta + i\sin\theta - 1$

$= -2\sin^2\dfrac{\theta}{2} + 2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$z - 2 = 2i\sin\dfrac{\theta}{2}\left[\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right]$

$z = 1 + \cos\theta + i\sin\theta$

$z = 2\cos\dfrac{\theta}{2}\left[\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right]$

$\dfrac{z-2}{z} = i\tan\dfrac{\theta}{2} = i\tan(\arg z)$

Sol.18 $(z+1)^7 = -z^7$

$|z+1|^7 = |z|^7 \quad \Rightarrow \quad |z+1| = |z|$

put $z = x + iy$

$(x+1)^2 + y^2 = x^2 + y^2 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

(a) $\sum_{r=1}^{7} \mathrm{Re}(z_r) = -\frac{7}{2}$

(b) $\sum_{r=1}^{7} \mathrm{Ing}(z_r) = 0$

Sol.19 on dividing by $z - i$ that is $z - i = 0 \Rightarrow z = i$

$f(i) = i$ ....(1) & $f(-i) = 1 + i$ ....(2)

since $z^2 + 1$ is a quadratic expression,

so on dividing $f(z)$ by $z^2 + 1$: remainder will be a linear expression so

$f(z) = g(z) . (z^2 + 1) + az + b$ ......(3)

$f(i) = ai + b = i$ ......(4)

& $f(-i) = -ai + b = 1 + i$ ......(5)

$\Rightarrow$ $a = i/2$ & $b = 1/2 + i$

so remainder $= az + b = \underbrace{1/2\, iz + 1/2 + i}$

Sol.20 $N = (a^3 - 3ab^2) + i\,[3a^2b - b^3 - 107]$

$3a^2b - b^3 - 107 = 0 \quad \Rightarrow 3a^2 b - b^3 = 107$

If $b = 1 \quad \Rightarrow \quad a = 6$

$N = 216 - 18 = 198$

Sol.21 $x^4 + 9x^3 + bx^2 + (x + d) = 0$

x_1, x_2, x_3, x_4

$x_1 + x_2 = 3 + 4i$

$x_3x_4 = 13 + i \Rightarrow x_1x_2 = 13 - i$

Root will be like

$\alpha + i\beta,\ \alpha - i\beta,\ \gamma + i\delta,\ \gamma - i\delta$

$x_1 \quad\quad x_3 \quad\quad x_2 \quad\quad x_4$

$x_1 + x_2 = 3 + 4i \ \Rightarrow \alpha + \delta = 3$

sum of the roots $= 2(\alpha + \delta) = 6$

$\Rightarrow x_3 + x_4 = -3 - 4i$

$b = \Sigma x_1x_2 = 26 + 25 = 51$

Sol.22 $f(z) = |z^3 - z + 2|$

put $z = \cos\theta + i\sin\theta$

$f(z) = |\cos 3\theta + i\sin 3\theta - \cos\theta - i\sin\theta + 2|$

$= \sqrt{(\cos 3\theta - \cos\theta + 2)^2 + (\sin 3\theta - \sin\theta)^2}$

now differentiate and get the value of θ

Sol.23 $az^2 + bz + c = 0$

$z_1 + z_2 = -b/a$

$z_1z_2 = c/a$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta = \tan^{-1}\left|\frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1y_2}{x_1x_2}}\right| \quad .....(1)$$

$(x_1 + x_2) + i\,(y_1 + y_2) = -b/a$

$\Rightarrow\ x_1 + x_2 = -b/a \ \ \&\ \ y_1 + y_2 = 0$

$z_1 z_2 = c/a$

$\Rightarrow\ x_1 x_1 - y_1 y_2 = c/a \ \ \&\ \ x_1 y_2 + y_1 x_2 = 0$

putting all the value in equation (1)

& get the answer

Sol.24 $(1 + i)\ (z^2)$

$= (1 + i)\,(x^2 - y^2 + 2ixy)$

$\mathrm{Re} = x^2 - y^2 - 2xy > 0$

$x^2 - y^2 - 2xy > 0$

this represent two perpendicular straight lines and draw the region.

Sol.25 $z^{2m} + z^{2m-1} + z^{2m-2} + ... + z + 1 = (z - z_1)(z - z_2)$ $.....(z - z_{2m})$

$\ell n\,(z^{2m} + z^{2m-1} + z^{2m-2} + ... + z + 1)$

$= \ell n\,(z - z_1) + \ell n(z - z_2) + ... + \ell n(z - z_{2m})$

Differentiate

$$\left[\frac{2m\, z^{2m-1} + (2m-1)\, z^{2m-2} + ... + 1}{z^{2m} + z^{2m-1} + ... + z + 1}\right]$$

$$= \frac{1}{z - z_1} + \frac{1}{z - z_2} + ... + \frac{1}{z - z_{2m}}$$

put $z = 1$

$$\frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \frac{1}{z_3 - 1} + ... + \frac{1}{z_{2m} - 1}$$

$$= -\left[\frac{2m + (2m-1) + ... + 1}{1 + 1 + 1 + ... + 1}\right] = -m$$

Sol.26 $\left[r_1\, e^{i\theta_1}\right]^n = \left[r_2\, e^{i\theta_2}\right]$ $\qquad r_1 = r_2 = 1$

$e^{i\theta_1} = e^{i\theta_2/n}$ $\qquad \theta_1 = 2\tan^{-1} x$

$\theta_1 = \frac{\theta_2}{n}$ $\qquad \theta_2 = 2\tan^{-1} a$

$2n\tan^{-1} x = 2\tan^{-1} a$

$\tan^{-1} x = \frac{1}{n}\tan^{-1} a \quad \Rightarrow$ x is always real